Bright-dark solitary-wave solutions of a multidimensional nonlinear Schrödinger equation

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An intrafield symbiotic form of the bright and the dark solitons, termed a *symbion*, is found in the framework of the Hartree approximation for a multidimensional nonlinear Schrödinger equation. Algebraically, the symbion can be expressed by the product of the two fields; its eigenvalue and stationary field distribution are determined analytically in a self-consistent fashion. The error due to the approximation is analyzed.

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As is well known, the canonical $(1+1)$-dimensional nonlinear Schrödinger equation (NLSE) has two kinds of soliton solutions: bright and dark solitons [1–3]. The former can exist for the case that $\kappa_{ds}\kappa_{nl}>0$, while the latter can exist for $\kappa_{ds}\kappa_{nl}<0$, where κ_{ds} and κ_{nl} are, respectively, the dispersion and the cubic nonlinear coefficients. For the $(2+1)$-dimensional NLSE, in addition to the two terms, effects due to the diffraction (κ_{df}) are incorporated as an additional term. Thus, depending on the sign combination of the three terms, there exist three cases: case (1) $\kappa_{df}\kappa_{ds}>0, \kappa_{df}\kappa_{nl}>0$; case (2) $\kappa_{df}\kappa_{ds}>0, \kappa_{df}\kappa_{nl}<0$; and case (3) $\kappa_{df}\kappa_{ds}<0$. Bright and dark stationary fields can be supported in cases (1) and (2), respectively [4]. Here, for the $(2+1)$ dimension the term "bright" or "dark" is used in space and time. For the last case [case (3)] the following questions arise: Does the case provide any stationary form of solitary waves? If so, how is it expressed algebraically? In this Brief Report we derive a solitary-wave solution for the case using a Hartree approximation of the multidimensional NLSE. Initially, this approach was applied to many-electron systems in quantum mechanics [5]. In recent years, it has been found so useful for obtaining stationary modes in optics [6–8] and mesoscopic quantum mechanics [9,10]. The particular solution we derive is a symbiotic form of the bright and the dark solitons; along one transverse dimension (x) the field profile has a bright form that is expressed by the hyperbolic-secant function, while along another transverse dimension (t) it has a dark (black) form that is expressed by the hyperbolic-tangent function. Generalization is made to the D-dimensional symbiotic form of a $(D+1)$-dimensional NLSE ($D=3,4,\ldots$). We find that for arbitrary dimension D a bright-dark symbion can be derived as a self-consistent solution of the D-dimensional Hartree approximation. To estimate the accuracy of the approximation, error analysis is performed through direct substitution of the self-consistent solution into the relevant NLSE.

We first consider a $(2+1)$-dimensional NLSE that belongs to case (3):

$$iq_z+\tfrac{1}{2}(\partial_x^2 q-\partial_t^2 q)+|q|^2q=0\ , \tag{1}$$

where q is the complex amplitude that represents an envelope of a relevant wave field, z is the longitudinal axis, and x and t are the two transverse axes (e.g., x and t represent, respectively, space and time), all of which are scaled appropriately ($\kappa_{df}=\frac{1}{2}, \kappa_{ds}=-\frac{1}{2}, \kappa_{nl}=1$).

As an ansatz of a stationary solution of Eq. (1) we set

$$q(z;x,t)=f(x)g(t)\exp(i\beta z)\ , \tag{2}$$

where f (bright) and g (dark) are shape functions for the x and t axes, respectively [$f(0)\neq 0, \lim_{|x|\to\infty} f(x)=0$, $g(0)=0$, and $\lim_{|t|\to\infty} d_t g(t)=0$], and β is the eigenvalue, which should be real. Below we shall examine whether the bright-dark symbion, i.e., an eigensolution, which meets all the requirements on the asymptotic behavior of the functions and on the eigenvalue, is obtainable analytically as a stationary solution of Eq. (1). Since no analytical method for obtaining the exact solution of the symbion is available, we must use an approximate approach. Among some candidates we find that the Hartree approximation (the self-consistent-field approximation) [5] is applicable to the present problem. This method was initially developed by Hartree as a solution method for many-electron problems that were encountered in quantum mechanics of atoms. Subsequently, it was refined by Fock and Slater to what we call the Hartree-Fock-Slater approximation in solid-state physics [11]. In recent years, this approach has been found so useful for obtaining stationary modes in optics [6–8] as well as wave functions of an electron in low-dimensional mesoscopic systems such as quantum wires [9] and boxes [10].

To determine the three unknowns (β,f,g) using the Hartree procedure, first we substitute Eq. (2) into Eq. (1):

$$\tfrac{1}{2}[(d_x^2 f)g-f d_t^2 g]+(|f|^2|g|^2-\beta)fg=0\ . \tag{3}$$

Multiplying this by g^* and integrating the product from $t=-\infty$ to ∞, we obtain an integrodifferential equation

$$\tfrac{1}{2}d_x^2 f+(\gamma_f|f|^2+\tfrac{1}{2}\delta_f-\beta)f=0\ , \tag{4}$$

with

$$\gamma_f=\int_{-\infty}^{\infty}|g|^4dt\Big/\int_{-\infty}^{\infty}|g|^2dt\ , \tag{5a}$$

$$\delta_f=\int_{-\infty}^{\infty}|d_t g|^2dt\Big/\int_{-\infty}^{\infty}|g|^2dt\ , \tag{5b}$$

where the asterisk denotes complex conjugate.

Similarly, multiplying Eq. (3) by f^* and integrating the

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product from $x=-\infty$ to ∞, we obtain

$$-\tfrac{1}{2}d_t^2 g+(\gamma_g|g|^2-\tfrac{1}{2}\delta_g-\beta)g=0\ , \tag{6}$$

with

$$\gamma_g=\int_{-\infty}^{\infty}|f|^4dx \Big/ \int_{-\infty}^{\infty}|f|^2dx\ , \tag{7a}$$

$$\delta_g=\int_{-\infty}^{\infty}|d_xf|^2dx \Big/ \int_{-\infty}^{\infty}|f|^2dx\ . \tag{7b}$$

Because Eqs. (4) and (6) are the canonical one-dimensional NLSE's, they allow, respectively, the bright and the dark soliton solutions [1–3]:

$$f(x)=\mathrm{sech}(\sqrt{\gamma_f}x)\ , \tag{8a}$$

with the eigenvalue

$$\beta=\tfrac{1}{2}(\gamma_f+\delta_f)\ , \tag{8b}$$

and

$$g(t)=\tanh(\sqrt{\gamma_g}t)\ , \tag{9a}$$

with the eigenvalue

$$\beta=\gamma_g-\tfrac{1}{2}\delta_g\ . \tag{9b}$$

Here, to render the solution of Eq. (2) self-consistent, Eqs. (8b) and (9b) must be coincident.

Substitution of Eqs. (8a) and (9a) into Eqs. (7) and (5), respectively, and doing the integrations yield

$$\gamma_f=1\ ,\quad \delta_f=0\ , \tag{10a}$$

$$\gamma_g=\tfrac{2}{3}\ ,\quad \delta_g=\tfrac{1}{3}\ . \tag{10b}$$

From Eqs. (8b), (9b), and (10) we finally obtain a self-consistent eigensolution:

$$f(x)=\mathrm{sech}x\ ,\quad g(t)=\tanh(\sqrt{\tfrac{2}{3}}t)\ , \tag{11a}$$

with

$$\beta=\tfrac{1}{2}\ . \tag{11b}$$

The result of Eq. (11a) with Eq. (2) indicates that the intensity profile of the total field $|q|^2$ exhibits a symbiotic form of the bright (f) and the dark (g) solitons. A bird's-eye view of the intensity of the symbion, $|q(x,t)|^2=(fg)^2$, is plotted in Fig. 1(a).

Because the self-consistent solution, Eq. (11), has been derived through the approximation, the inclusion of error

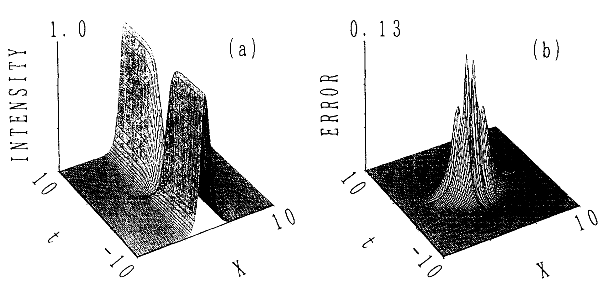

FIG. 1. Bird's-eye plots of (a) the intensity $(fg)^2$ [Eq. (11a); Eqs. (18b) and (18c) with $D=2$] and (b) the error distribution $|E(x,t)|$ [Eq. (12)] of the two-dimensional ($D=2$) bright-dark symbion. $\max|E|=0.128$ at $(x,t)=(0,\pm 0.807)$. The frame is chosen to be $|x|\le 10$, $|t|\le 10$.

should be elucidated. Below we estimate the error E that arises due to the approximation by directly substituting the solution into the left-hand side of Eq. (3). After simple algebra it can be reduced to a compact form

$$E(x,t)=f(\tfrac{2}{3}-f^2)g(1-g^2)\ , \tag{12}$$

the magnitude of which is shown in Fig. 1(b). The maximum of $|E|$ is obtainable analytically; we find that $\max|E|=0.128$ at $(x,t)=(0,\pm 0.807)$. If the solution we have derived were exact, one would find that $E(x,t)\equiv 0$ over the entire region. Otherwise, the error comes from the separation-of-variable ansatz of Eq. (2) and the subsequent averaging algebra of Eqs. (4) and (6). Similar discussion will be applicable to Eq. (16) that follows.

In what follows we shall extend the approach presented above to higher-dimensional NLSE's. Besides the $(2+1)$-dimensional NLSE given by Eq. (1), such a physical system as described by the $(3+1)$-dimensional NLSE has recently been addressed [12,13]:

$$iq_z+\tfrac{1}{2}(\partial_x^2q+\partial_y^2q-\partial_t^2q)+|q|^2q=0\ , \tag{13}$$

where y is an additional transverse dimension. Only recently, numerical integrations of the similar equation to this have been performed to investigate spatiotemporal dynamics of a self-focusing ultrashort pulse in a normally dispersive medium [12,13].

Assuming the ansatz of a stationary solution of Eq. (13) in the form

$$q(z;x,y,t)=f(x)g(y)h(t)\exp(i\beta z)\ , \tag{14}$$

after the same procedure as taken in the $(2+1)$ dimension, we finally arrive at the self-consistent solution

$$f(x)=\mathrm{sech}(\sqrt{\tfrac{2}{3}}x)\ , \tag{15a}$$

$$g(x)=\mathrm{sech}(\sqrt{\tfrac{2}{3}}y)\ , \tag{15b}$$

$$h(t)=\tanh(\tfrac{2}{3}t)\ , \tag{15c}$$

$$\beta=\tfrac{2}{9}\ . \tag{15d}$$

A comparison of these results with those presented in Eq. (11) shows that the confinement of the field gets weaker (larger variance of the field profile and smaller eigenvalue) with increasing the number of transverse dimensions. The local error that is involved is written as

$$E(x,y,t)=fgh[\tfrac{8}{9}-\tfrac{2}{3}(f^2+g^2+\tfrac{2}{3}h^2)+(fgh)^2]\ . \tag{16}$$

We find analytically that $\max|E|=0.153$ at $(x,y,t)=(0,0,\pm 0.857)$, which is slightly larger than that in the $(2+1)$-dimensional case.

In general, for the $(D+1)$-dimensional NLSE,

$$iq_z+\tfrac{1}{2}(\partial_{x_1}^2+\partial_{x_2}^2+\cdots+\partial_{x_{D-1}}^2-\partial_t^2)q+|q|^2q=0\ , \tag{17}$$

we find that a D-dimensional symbion is possible; its eigensolution is given by

$$q(z;x_1,x_2,\ldots,x_{D-1},t)=f_1(x_1)f_2(x_2)\cdots f_{D-1}(x_{D-1})f_D(t)\exp(i\beta z)\ , \tag{18a}$$

$$f_j(x_j)=\mathrm{sech}[(\tfrac{2}{3})^{(D-2)/2}x_j] \quad \text{for } j=1,2,\ldots,D-1\ , \tag{18b}$$

$$f_D(t)=\tanh[(\tfrac{2}{3})^{(D-1)/2}t]\ , \tag{18c}$$

$$\beta=(\tfrac{1}{6})(\tfrac{2}{3})^{D-2}(5-D) \tag{18d}$$

for $D=2,3,4,\ldots$; for the globally defined coordinate in real space and time, the value of D will be restricted to 2 and 3, whereas for such locally defined coordinate systems as employed in many-particle systems [5,11] or in a phase space, no restriction will be placed on the upper limit of D. For the symbion with arbitrary dimension, such as those for $D=2$ and 3, which have been presented above, the error due to the Hartree approximation can be estimated as well in an analytical fashion. The results will be detailed elsewhere.

For lack of the exact solution being available, one cannot evaluate exactly the influence of the error on accuracy of the eigenvalue β. However, we can infer the effect through application of the Hartree method ($\beta_H, \max|E|$) to the canonical bright-field problem for which very accurate variational solution (β_v) is available [14]. The results are summarized as follows: $(D,\beta_v,\beta_H,\max|E|)=(2,0.191,0.222,0.111)$, $(3,0.0424,0.0741,0.259)$, and $(4,0,0,0.296)$, which suggest that the accuracy of the Hartree method is reasonable. Here the point that gives $\max|E|$ has been the origin (the center of field), irrespective of the dimension.

In addition, we consider the $(D+1)$-dimensional NLSE with negative (self-defocusing) nonlinearity:

$$iq_z+\tfrac{1}{2}(\partial^2_{x_1}+\partial^2_{x_2}+\cdots+\partial^2_{x_{D-1}}-\partial^2_t)q-|q|^2q=0\ . \tag{19}$$

Comparing this with Eq. (17), one may expect to obtain a symbion in the form of Eq. (18a), but the components being interchanged by

$$f_j(x_j)=\tanh(\alpha_j x_j) \quad \text{for } j=1,2,\ldots,D-1\ , \tag{20a}$$

$$f_D(t)=\mathrm{sech}(\alpha_D t)\ , \tag{20b}$$

which exhibit a dual form with Eqs. (18b) and (18c). Here α_j $(j=1,2,\ldots,D)$ represents a variance (a reciprocal spot size) to be determined in a self-consistent fashion, if a self-consistent value exists. Through the analysis, however, we have verified that no self-consistent eigenvalue is derivable for any dimension, indicating that Eq. (19) has no symbiotic solution at least in the framework of the Hartree approximation. This result is in marked contrast with that found for Eq. (17).

Finally, the possibility of experimentally observing evidence for the bright-dark symbion given in general by Eq. (18) should be mentioned. Aside from $D\geq4$, for $D=2$, and 3, the most promising physical system to observe it will be the focused-laser-pulse propagation in a Kerr-law optical medium. Requirements for the medium are the good transparency at least in the vicinity of the center wavelength of a laser, the fast response of the nonlinearity, and the high damage threshold against radiation. With these in mind, we would recommend the use of high-index glass or a highly nonlinear organic material such as 2,4-hexadyne-1, 6-diol (PTS), and 4-(N,N-dimethylamino)-3-acetamidonitrobenzene (DAN). As found from Eqs. (11) and (15), the symbion is bright in space (x,y) and dark in time (t). Practically, the strict realization of the latter is impossible since infinite energy is required for realizing infinite background of the hyperbolic tangent profile. However, this difficulty will not be crucial because through numerical simulations it was found that the dark soliton can be maintained even in a finite background provided that the width of the background pulse is large enough to ignore the effect of dispersion [15,16]. Another issue we should mention here is the stability problem of the symbion, which may be anticipated particularly for a higher dimension. This problem is beyond the scope of this paper, and will be elucidated elsewhere.

Recently, bright-dark solitary-wave pairs were reported for the case of nonlinear interaction of two optical waves, which result from the cross-phase modulation (the intermode coupling) [17–21] or the stimulated Raman scattering and loss [22]. In these configurations, two optical fields at different polarizations, modes, or wavelengths are incident simultaneously at the nonlinear medium, indicating that the interfield coupling is necessary to observe the pairs. Evidently, the situation is essentially different from the symbion we have found here where only a single optical field propagates through the nonlinear medium.

In conclusion, we have obtained an intrafield symbiotic form of the bright and the dark solitons using a Hartree approximation for a multidimensional NLSE. The error due to the approximation has been analyzed. The results presented herein may have relevance in diverse areas of science where wave evolution is described by the multidimensional NLSE that includes both space and time. Finally, we would address that the present report is open ended. One of several interesting open theoretical questions is whether the symbion is stable during propagation. More detailed theoretical and experimental investigations should be under way and will be reported elsewhere.

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[1] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[2] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[3] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 171 (1973).
[4] J. J. Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).
[5] D. R. Hartree, *The Calculation of Atomic Structures* (Wi-

ley, New York, 1957).
[6] D. A. Roberts and M. S. Stern, Electron. Lett. **23**, 784 (1987).
[7] K. Yanagawa, K. Hayata, and M. Koshiba, Trans. Inst. Electron. Inf. Commun. Eng. **J75-C-I**, 27 (1992).
[8] K. Hayata, Y. Tsuji, and M. Koshiba, J. Appl. Phys. **71**, 5266 (1992).
[9] Y. Tsuji, K. Hayata, and M. Koshiba, Trans. Inst. Electron. Inf. Commun. Eng. **J74-C-I**, 494 (1991).
[10] Y. Tsuji, K. Hayata, and M. Koshiba, Trans. Inst. Electron. Inf. Commun. Eng. **J75-C-I**, 542 (1992).
[11] J. C. Slater, *Quantum Theory of Atomic Structure* (McGraw-Hill, New York, 1960), Vol. 2.
[12] P. Chernev and V. Petrov, Opt. Lett. **17**, 172 (1992).
[13] J. E. Rothenberg, Opt. Lett. **17**, 583 (1992).
[14] K. Hayata and M. Koshiba, Opt. Lett. **17**, 841 (1992).
[15] D. Krökel, N. J. Halas, G. Giuliani, and D. Grischkowsky, Phys. Rev. Lett. **60**, 29 (1988).
[16] W. J. Tomlinson, R. J. Hawkins, A. M. Weiner, J. P. Heritage, and R. N. Thurston, J. Opt. Soc. Am. B **6**, 329 (1989).
[17] V. V. Afanasyev, Yu. S. Kivshar, V. V. Konotop, and V. N. Serkin, Opt. Lett. **14**, 805 (1989).
[18] V. V. Afanasjev, E. M. Dianov, and V. N. Serkin, IEEE J. Quantum Electron. **25**, 2656 (1989).
[19] L. Wang and C. C. Yang, Opt. Lett. **15**, 474 (1990).
[20] Yu. S. Kivshar, Opt. Lett. **17**, 1322 (1992).
[21] M. Shalaby and A. J. Barthelemy, IEEE J. Quantum Electron. **28**, 2736 (1992).
[22] R. I. Joseph and D. N. Christodoulides, Opt. Lett. **17**, 1101 (1992).